Orthogonal Projection

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Reference

- Textbook: Chapter 7.3, 7.4

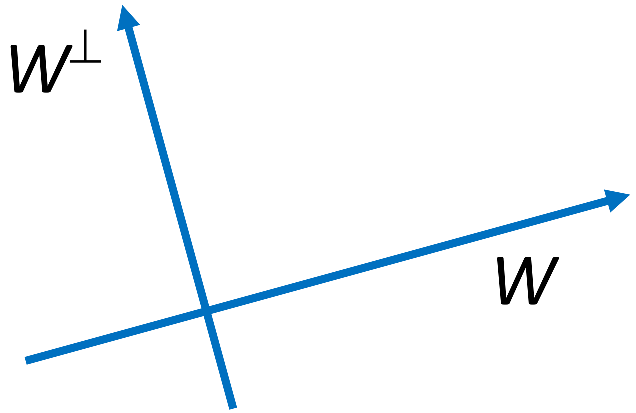
Orthogonal Projection

- What is Orthogonal Complement
- What is Orthogonal Projection
- How to do Orthogonal Projection
- Application of Orthogonal Projection

Orthogonal Complement

- The orthogonal complement of a nonempty vector set $\mathcal{S}$ is denoted as $\mathcal{S}^{\perp}$ (S perp).
- $\mathcal{S}^{\perp}$ is the set of vectors that are orthogonal to every vector in $\mathcal{S}$

$$S^{\perp} = \{v: v \cdot u = 0, \forall u \in S\}$$



$\mathcal{S} = \mathcal{R}^{n}$

$\mathcal{S} = \{0\}$

Orthogonal Complement

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- $\boldsymbol{S}^{\perp}$ is the set of vectors that are orthogonal to every vector in $\boldsymbol{S}$

$$S^{\perp} = \{v : v \cdot u = 0, \forall u \in S\}$$

$$W = \left\{ \begin{bmatrix} w_1 \\ w_2 \\ 0 \end{bmatrix} | w_1, w_2 \in \mathcal{R} \right\}$$

$$V = \left\{ \begin{bmatrix} 0 \\ 0 \\ v_3 \end{bmatrix} | v_3 \in \mathcal{R} \right\} = W^{\perp}?$$

$V \subseteq W^{\perp}$:

for all $\mathbf{v} \in V$ and $\mathbf{w} \in W$, $\mathbf{v} \bullet \mathbf{w} = 0$

$W^{\perp} \subseteq V$:

since $\mathbf{e}_1, \mathbf{e}_2 \in W$, all $\mathbf{z} = [\ z_1 \ \ z_2 \ \ z_3\ ]^T \in W^{\perp}$ must have $z_1 = z_2 = 0$

Properties of Orthogonal Complement

Is $S^{\perp}$ always a subspace?

For any nonempty vector set S, $(Span\ S)^{\perp} = S^{\perp}$

Let W be a subspace, and B be a basis of W.

$\Rightarrow$ $B^{\perp} = W^{\perp}$

What is $S \cap S^{\perp}$? $\Rightarrow$ Zero vector

Properties of Orthogonal Complement

- Example:

For W = Span{$\mathbf{u}_1$, $\mathbf{u}_2$}, where $\mathbf{u}_1 = [\ 1\ \ 1\ \ -1\ \ 4\ ]^T$ and $\mathbf{u}_2 = [\ 1\ -1\ \ 1\ \ 2\ ]^T$

$\mathbf{v} \in W^{\perp}$ if and only if $\mathbf{u}_1 \bullet \mathbf{v} = \mathbf{u}_2 \bullet \mathbf{v} = \mathbf{0}$

i.e., $\mathbf{v} = [\ x_1\ \ x_2\ \ x_3\ \ x_4\ ]^T$ satisfies

$$\begin{aligned} x_1 + x_2 - x_3 + 4x_4 &= 0 \\ x_1 - x_2 + x_3 + 2x_4 &= 0. \end{aligned} \quad \Leftrightarrow \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3x_4 \\ x_3 - x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

$\Leftrightarrow \mathcal{B} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for $W^{\perp}$.

$$A = \begin{bmatrix} 1 & 1 & -1 & 4 \\ 1 & -1 & 1 & 2 \end{bmatrix}$$

$W^{\perp}$ = Solutions of "Ax=0" = Null A

Properties of Orthogonal Complement

- For any matrix A

$$(Row\ A)^{\perp} = Null\ A$$

$\mathbf{v} \in (\text{Row } A)^{\perp}$

$\Leftrightarrow A\mathbf{v} = \mathbf{0}.$

$$(Col\ A)^{\perp} = Null\ A^{T}$$

$(\text{Col } A)^{\perp} = (\text{Row } A^{T})^{\perp} = \text{Null } A^{T}.$

For any subspace W of R^n

$$dimW + dimW^{\perp} = n$$

Unique

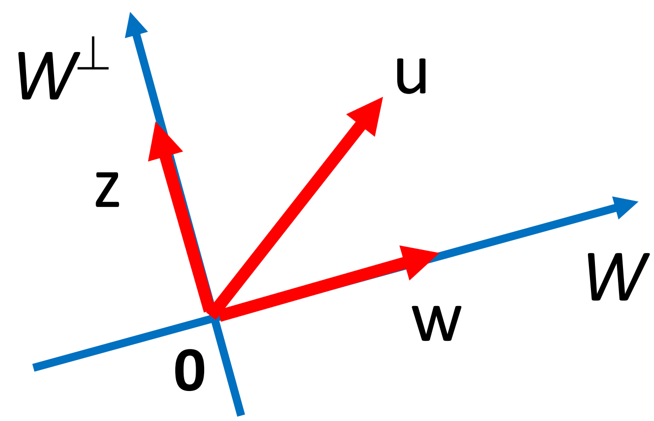
For any subspace W of R^n $\quad dimW + dimW^{\perp} = n$

$$\underbrace{\text{Basis: } \{w_1, w_2, \cdots, w_k\} \quad \text{Basis: } \{z_1, z_2, \cdots, z_{n-k}\}}_{\text{Basis for } R^n}$$

For ***every*** vector u,

u = w + z ***(unique)***

$\in W$ $\quad \in W^{\perp}$



Orthogonal Projection

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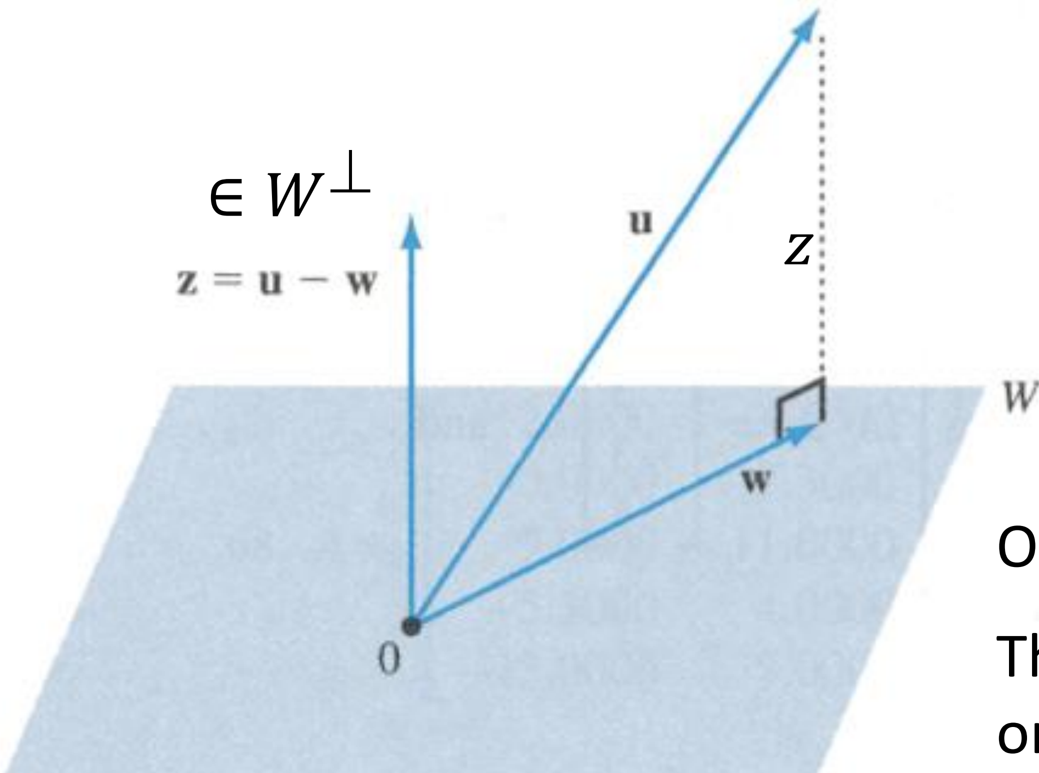
Orthogonal Projection



orthogonal projection

u = w + z ***(unique)***

$\in W$

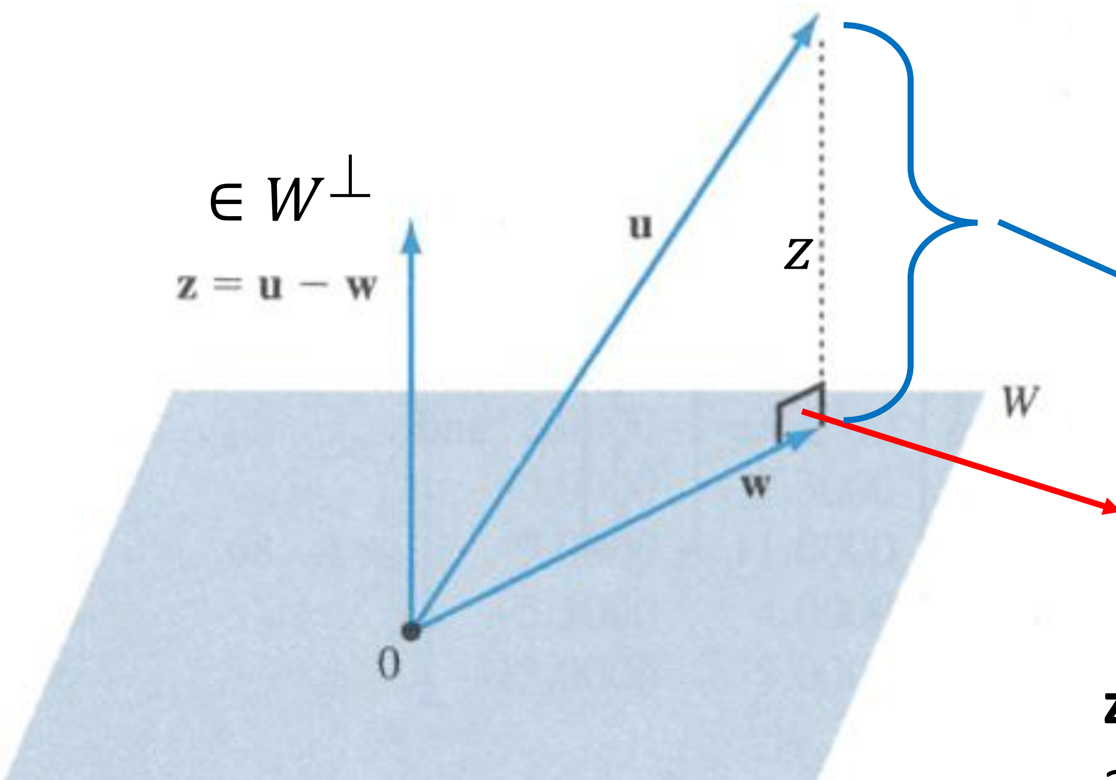
$\in W^{\perp}$

Orthogonal Projection Operator:

The function $U_W(u)$ is the orthogonal projection of u on W.

Linear?

Orthogonal Projection



w in subspace W is "closest" to vector **u**.

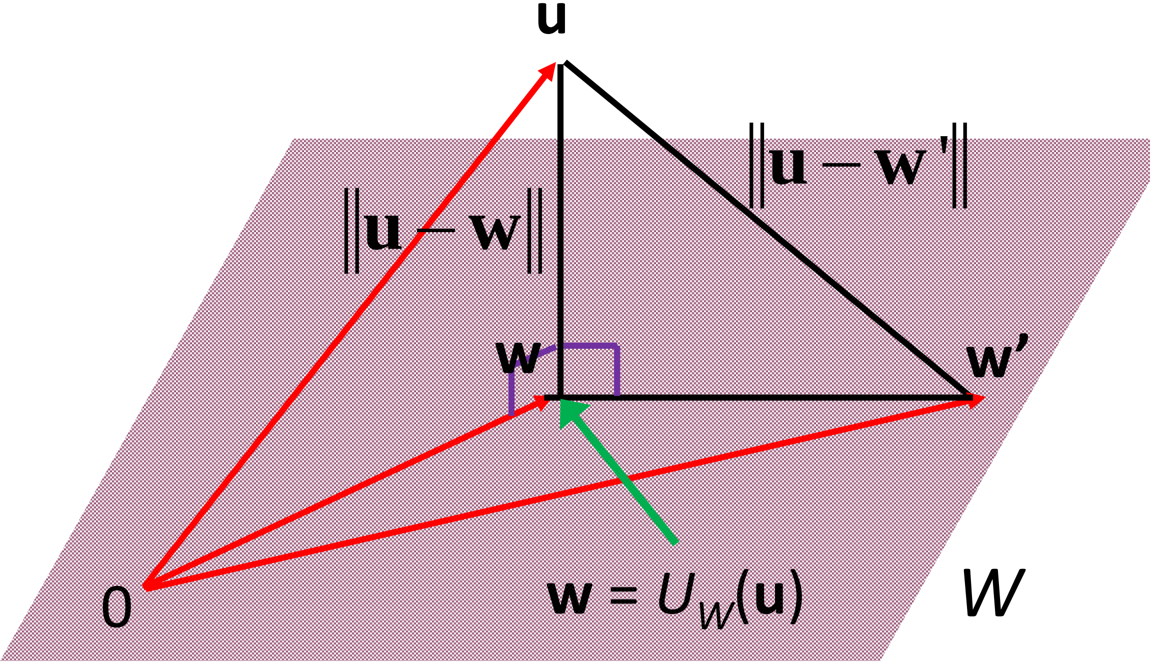
"closest"

Always orthogonal

z is always orthogonal to all vectors in W.

Closest Vector Property

- Among all vectors in subspace W, the vector closest to u is the orthogonal projection of u on W



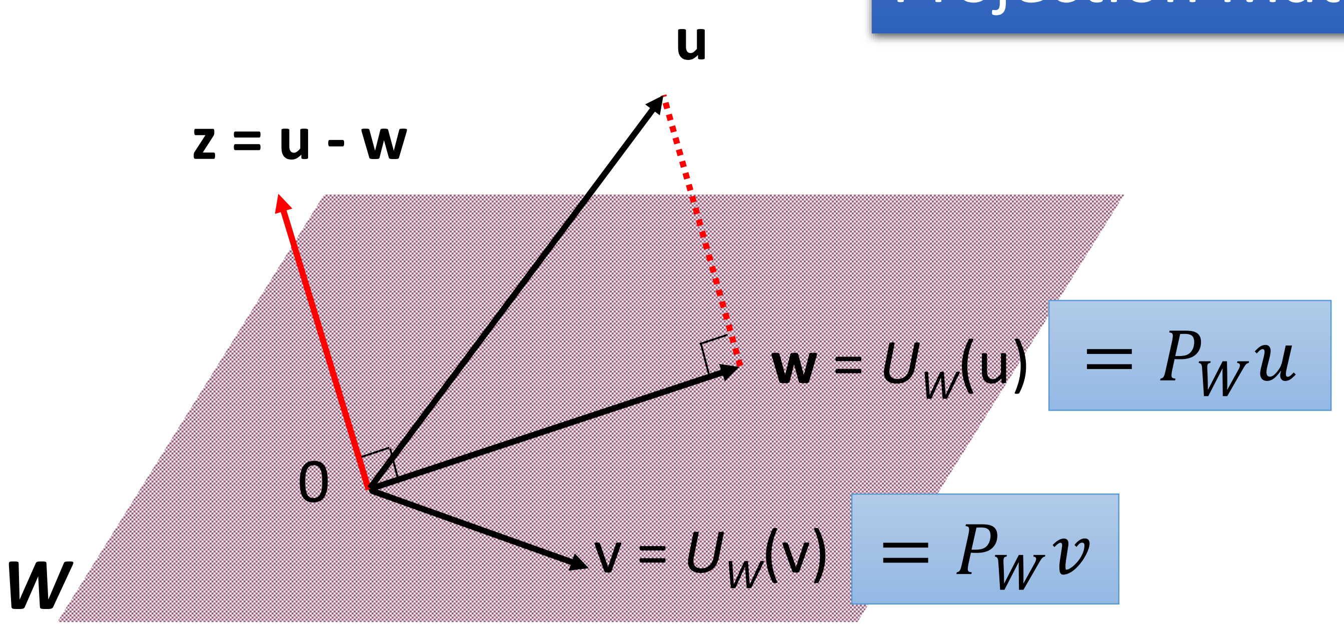
$\forall\ \mathbf{w}, \mathbf{w}', \mathbf{w} - \mathbf{w}' \in W.$

The distance from a vector u to a subspace W is the distance between u and the orthogonal projection of u on W

Orthogonal Projection Matrix

Orthogonal projection operator is linear.

→ It has standard matrix. → Orthogonal Projection Matrix P_W



Orthogonal Projection

What is Orthogonal Complement

What is Orthogonal Projection

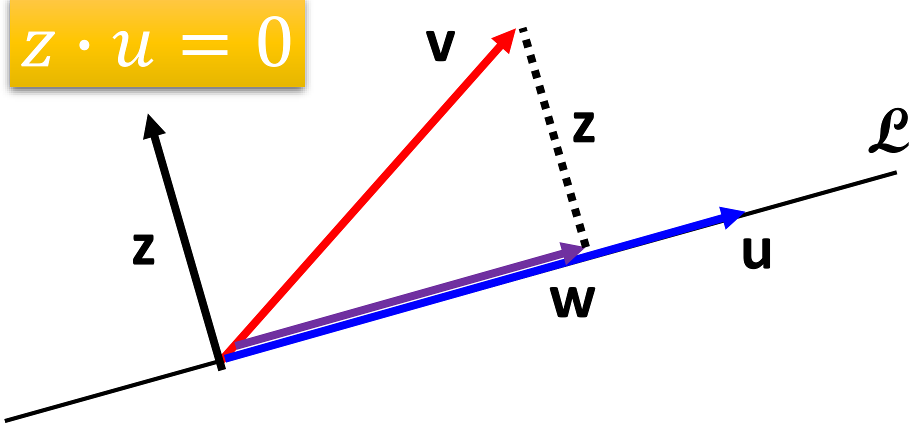
How to do Orthogonal Projection

Application of Orthogonal Projection

Orthogonal Projection on a line

- Orthogonal projection of a vector on a line

$z \cdot u = 0$



v: any vector

u: any nonzero vector on $\mathcal{L}$

w: orthogonal projection of **v** onto $\mathcal{L}$, **w** = c**u**

z: **v** − **w**

$$(v - w) \cdot u \;= (v - cu) \cdot u \;= v \cdot u - cu \cdot u \;= v \cdot u - c\|u\|^2$$

=0

$$c = \frac{v \cdot u}{\|u\|^2} \qquad w = cu = \frac{v \cdot u}{\|u\|^2} u$$

Distance from tip of **v** to $\mathcal{L}$: $\|z\| = \|v - w\| = \left\| v - \frac{v \cdot u}{\|u\|^2} u \right\|$

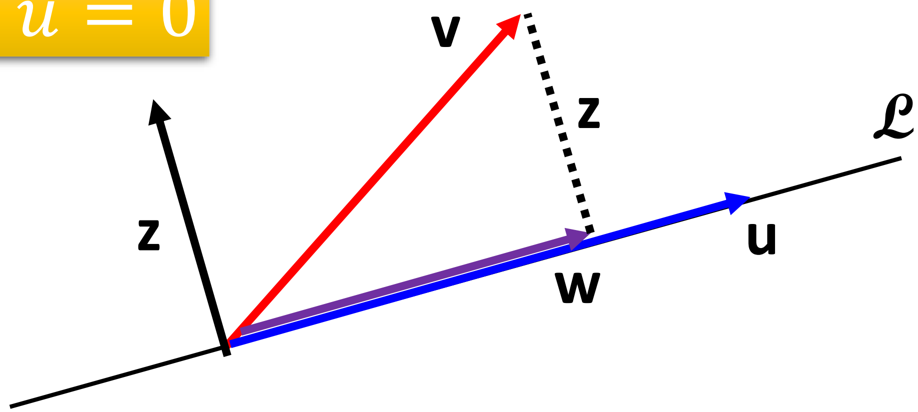
Orthogonal Projection

$$c = \frac{v \cdot u}{\|u\|^2}$$

$$w = cu = \frac{v \cdot u}{\|u\|^2} u$$

- Example:

$$z \cdot u = 0$$



$\mathcal{L}$ is $y = (1/2)x$

$$v = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \quad u = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Orthogonal Projection Matrix

- Let C be an n x k matrix whose columns form a basis for a subspace W

$$P_W = C(C^TC)^{-1}C^T$$

n x n

Proof: Let $\mathbf{u} \in \mathcal{R}^n$ and $\mathbf{w} = U_W(\mathbf{u})$.

Since $W = \text{Col } C$, $\mathbf{w} = C\mathbf{b}$ for some $\mathbf{b} \in \mathcal{R}^k$

and $\mathbf{u} - \mathbf{w} \in W^\perp$

$\Rightarrow \mathbf{0} = C^T(\mathbf{u} - \mathbf{w}) = C^T\mathbf{u} - C^T\mathbf{w} = C^T\mathbf{u} - C^TC\mathbf{b}.$

$\Rightarrow C^T\mathbf{u} = C^TC\mathbf{b}.$

$\Rightarrow \mathbf{b} = (C^TC)^{-1}C^T\mathbf{u}$ and $\mathbf{w} = C(C^TC)^{-1}C^T\mathbf{u}$ as C^TC is invertible.

Orthogonal Projection Matrix

- Let C be an n x k matrix whose columns form a basis for a subspace W

$$P_W = C(C^T C)^{-1} C^T$$

n x n

Let C be a matrix with linearly independent columns. Then $C^T C$ is invertible.

Proof: We want to prove that $C^T C$ has independent columns.

Suppose $C^T C\mathbf{b} = \mathbf{0}$ for some $\mathbf{b}$.

$\Rightarrow \mathbf{b}^T C^T C\mathbf{b} = (C\mathbf{b})^T C\mathbf{b} = (C\mathbf{b}) \bullet (C\mathbf{b}) = \|C\mathbf{b}\|^2 = 0.$

$\Rightarrow \; C\mathbf{b} = \mathbf{0} \; \Rightarrow \; \mathbf{b} = \mathbf{0}$ since C has L.I. columns.

Thus $C^T C$ is invertible.

Orthogonal Projection Matrix

- Example: Let *W* be the 2-dimensional subspace of $\mathcal{R}^3$ with equation $x_1 - x_2 + 2x_3 = 0$.

$$P_W = C(C^T C)^{-1} C^T$$

W has a basis $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$ $\quad C = \begin{bmatrix} 1 & -2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$P_W = \frac{1}{6} \begin{bmatrix} 5 & 1 & -2 \\ 1 & 5 & 2 \\ -2 & 2 & 2 \end{bmatrix} \qquad P_W \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}$$

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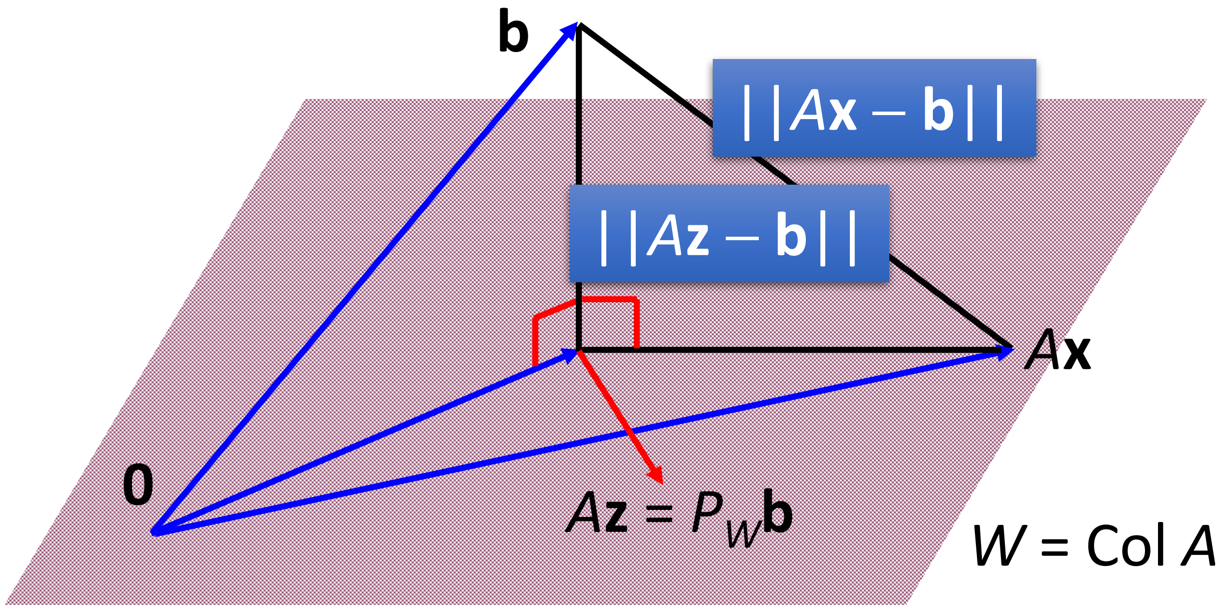
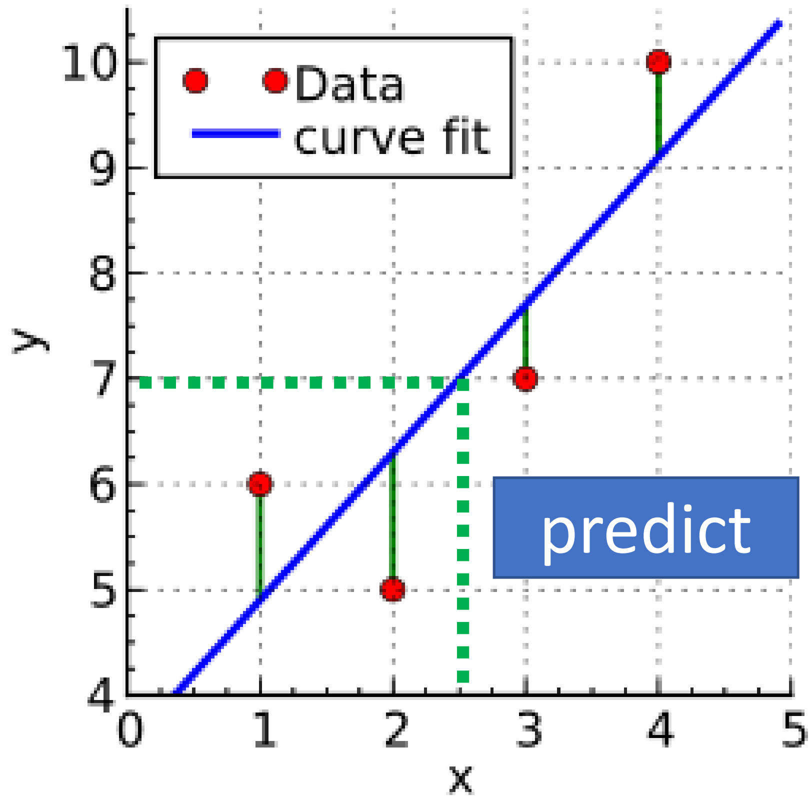
Solution of Inconsistent System of Linear Equations

- Suppose $A\mathbf{x} = \mathbf{b}$ is an inconsistent system of linear equations.
- $\mathbf{b}$ is not in the column space of A
- Find vector $\mathbf{z}$ minimizing $||A\mathbf{z} - \mathbf{b}||$



Least Square Approximation



data pairs:

$x_1 \rightarrow y_1$

$x_2 \rightarrow y_2$

$\vdots$

$x_i \rightarrow y_i$

$\vdots$

e.g.

(今天股票,明天股票)

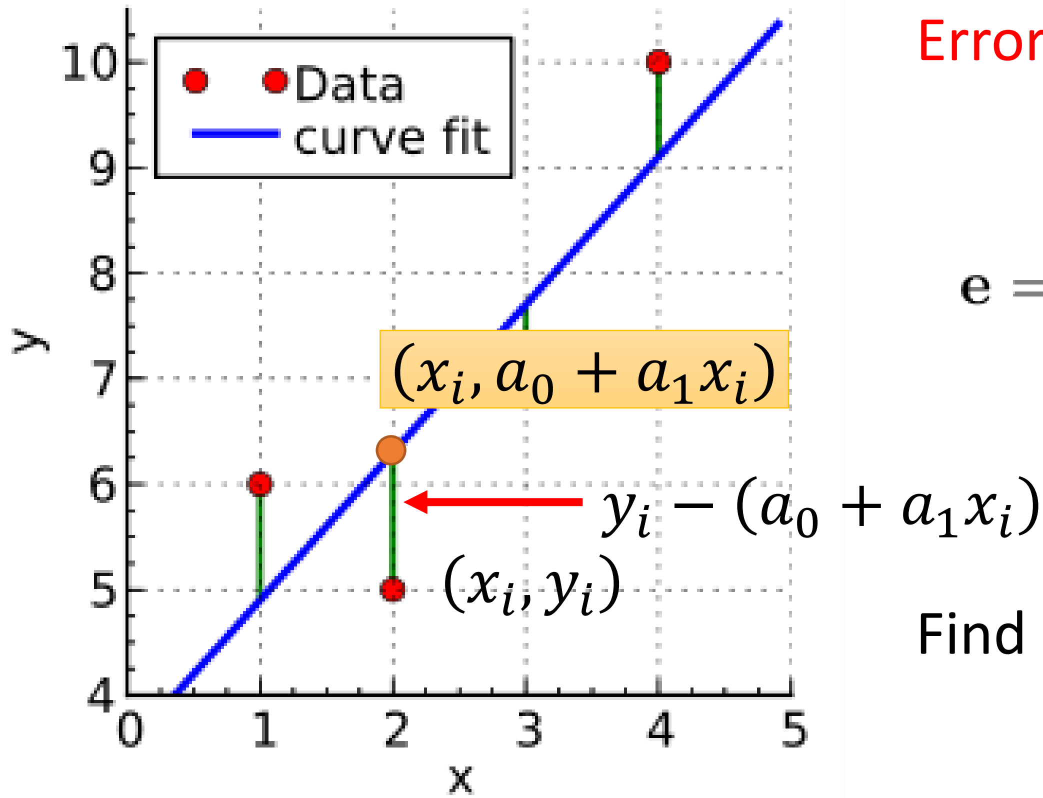
(今天PM2.5,明天PM2.5)

Find the “least-square line” $y = a_0 + a_1 x$ to best fit the data

Regression

Least Square Approximation

$y = a_0 + a_1 x$



Error Vector:

$$\mathbf{e} = \begin{bmatrix} y_1 - (a_0 + a_1 x_1) \\ y_2 - (a_0 + a_1 x_2) \\ \vdots \\ y_n - (a_0 + a_1 x_n) \end{bmatrix}$$

Find a_0 and a_1 minimizing E

$$E = ||\mathbf{e}||^2$$

$$E = [y_1 - (a_0 + a_1 x_1)]^2 + [y_2 - (a_0 + a_1 x_2)]^2 + \cdots + [y_n - (a_0 + a_1 x_n)]^2$$

Least Square Approximation

Error Vector:

$$\mathbf{e} = \left[\begin{array}{c} \boxed{y_1} - \boxed{(a_0} + \boxed{a_1 x_1)} \\ y_2 - (a_0 + a_1 x_2) \\ \vdots \\ y_n - (a_0 + a_1 x_n) \end{array} \right]$$

$$\mathbf{e} = \underline{\mathbf{y}} - \underline{a_0 \mathbf{v}_1} - \underline{a_1 \mathbf{v}_2}$$

$$\boxed{\mathbf{y} \triangleq \left[\begin{array}{c} y_1 \\ y_2 \\ \vdots \\ y_n \end{array} \right]}, \boxed{\mathbf{v}_1 \triangleq \left[\begin{array}{c} 1 \\ 1 \\ \vdots \\ 1 \end{array} \right]}, \boxed{\mathbf{v}_2 \triangleq \left[\begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_n \end{array} \right]}$$

Find a_0 and a_1 minimizing E

$$E = ||\mathbf{e}||^2$$

$$C \triangleq \left[\begin{array}{cc} \mathbf{v}_1 & \mathbf{v}_2 \end{array} \right], \text{ and } \mathbf{a} = \left[\begin{array}{c} a_0 \\ a_1 \end{array} \right]$$

$$E = ||\mathbf{y} - (a_0\mathbf{v}_1 + a_1\mathbf{v}_2)||^2 = ||\mathbf{y} - C\mathbf{a}||^2$$

Least Square Approximation

Find **a** minimizing

$$E = \|\mathbf{y} - C\mathbf{a}\|^2$$

$\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ (L.I.)

$$\mathbf{y} \triangleq \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \mathbf{v}_1 \triangleq \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \mathbf{v}_2 \triangleq \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Ca is the orthogonal projection of **y** on W = Span $\mathcal{B}$.

$$C \triangleq \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}, \text{ and } \mathbf{a} = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}$$

find **a** such that $C\mathbf{a} = P_W\mathbf{y}$

$$\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = (C^TC)^{-1}C^T\mathbf{y}$$

Example 1

Rough weight x_i (in pounds)	Finished weight y_i (in pounds)
2.60	2.00
2.72	2.10
2.75	2.10
2.67	2.03
2.68	2.04

$$C = \begin{bmatrix} 1 & 2.60 \\ 1 & 2.72 \\ 1 & 2.75 \\ 1 & 2.67 \\ 1 & 2.68 \end{bmatrix} \quad \mathbf{y} = \begin{bmatrix} 2.00 \\ 2.10 \\ 2.10 \\ 2.03 \\ 2.04 \end{bmatrix}$$

$$\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = (C^T C)^{-1} C^T \mathbf{y} \approx \begin{bmatrix} 0.056 \\ 0.745 \end{bmatrix}$$

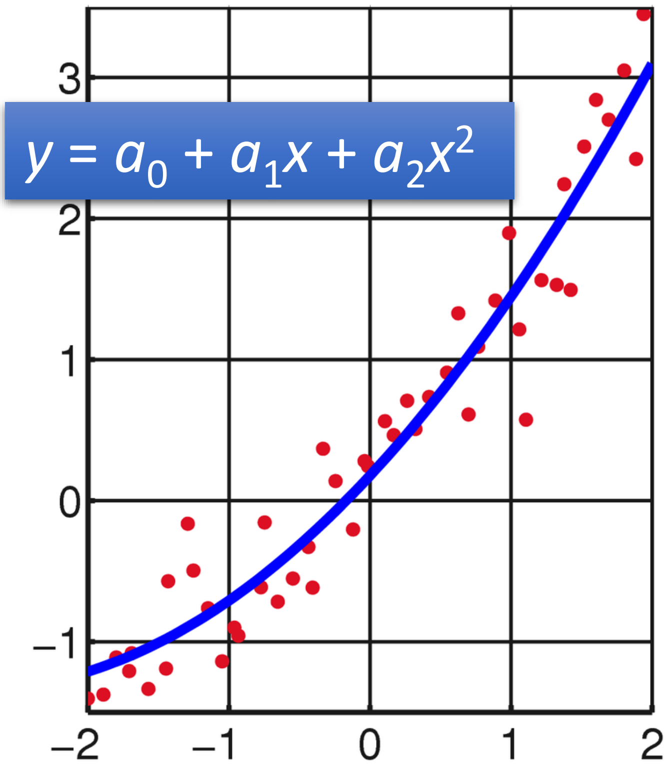
$\Rightarrow y = 0.056 + 0.745x.$

Prediction:
if the rough weight is 2.65,
the finished weight is
0.056 +0.745(2.65) = 2.030.

(estimation)

Least Square Approximation

- Best quadratic fit: using $y = a_0 + a_1x + a_2x^2$ to fit the data points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$



$$e = \begin{bmatrix} y_1 - (a_0 + a_1x_1 + a_2x_1^2) \\ y_2 - (a_0 + a_1x_2 + a_2x_2^2) \\ \vdots \\ y_n - (a_0 + a_1x_n + a_2x_n^2) \end{bmatrix}$$

Find a_0, a_1 and a_2 minimizing E

$$E = ||\mathbf{e}||^2$$

Least Square Approximation

- Best quadratic fit: using $y = a_0 + a_1 x + a_2 x^2$ to fit the data points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} x_1^2 \\ x_2^2 \\ \vdots \\ x_n^n \end{bmatrix}$$

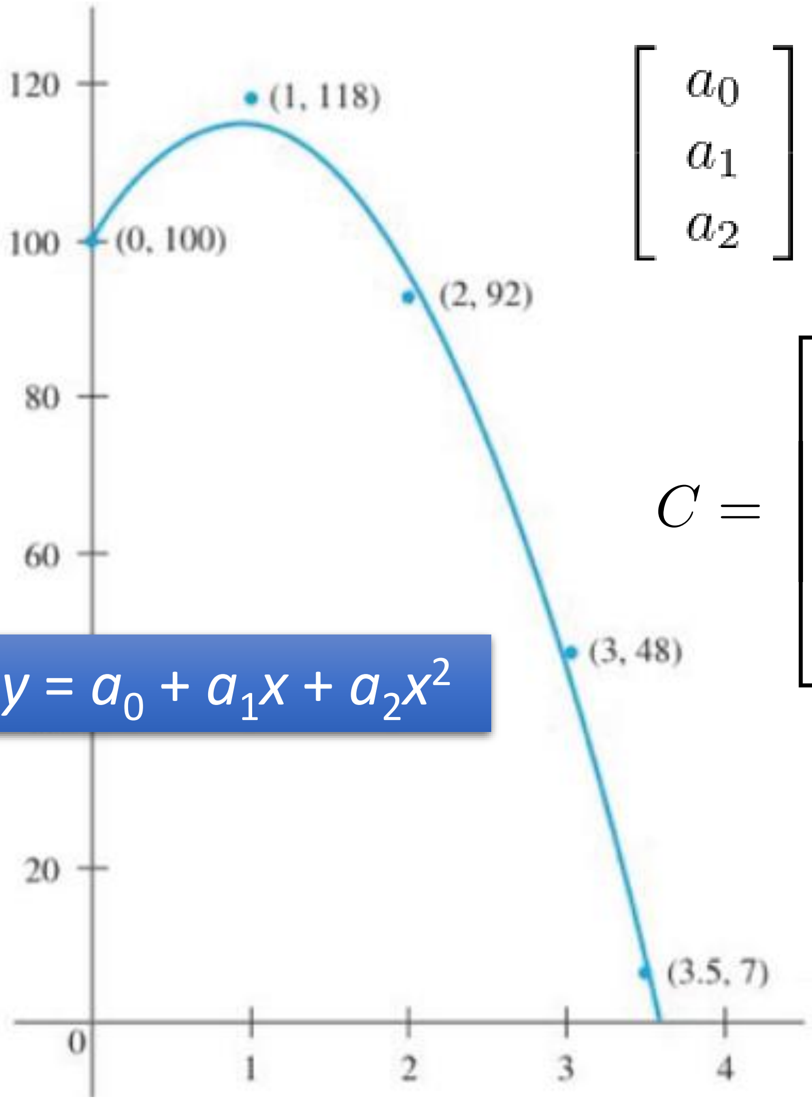
$$e = \begin{bmatrix} y_1 - (a_0 + a_1 x_1 + a_2 x_1^2) \\ y_2 - (a_0 + a_1 x_2 + a_2 x_2^2) \\ \vdots \\ y_n - (a_0 + a_1 x_n + a_2 x_n^2) \end{bmatrix}$$

$$C = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix}$$

Find a_0, a_1 and a_2 minimizing E

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = (C^T C)^{-1} C^T \mathbf{y}.$$

$$E = ||\mathbf{e}||^2$$



$y = a_0 + a_1x + a_2x^2$

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = (C^TC)^{-1}C^T\mathbf{y}$$

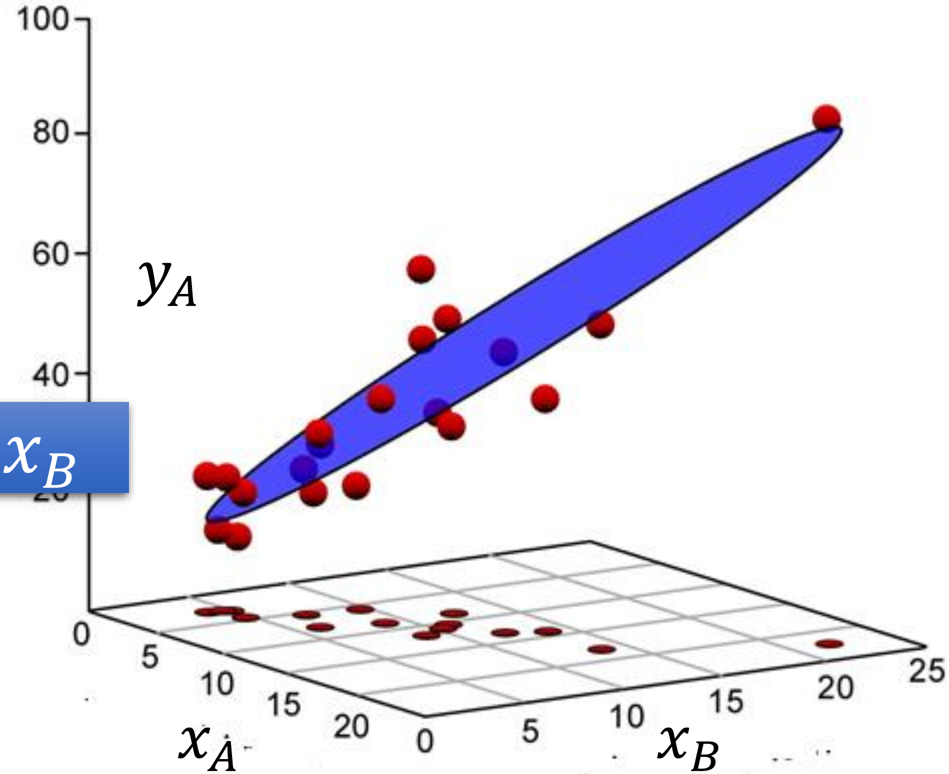
$$C = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 3.5 & 12.25 \end{bmatrix} \quad \mathbf{y} = \begin{bmatrix} 100 \\ 118 \\ 92 \\ 48 \\ 7 \end{bmatrix}$$

$$y = 101.00 + 29.77x - 16.11x^2$$

Best fitting polynomial of any desired maximum degree may be found with the same method.

Multivariable Least Square Approximation

$$y_A = a_0 + a_1 x_A + a_2 x_B$$



http://www.palass.org/publications/newsletter/palaeomath-101/palaeomath-part-4-regression-iv